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Reliability Estimation and Design with Insufficient Data Based on Possibility Theory

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Early in the engineering design cycle, it is difficult to quantify product reliability or compliance to performance targets because of insufficient data or information for modeling the uncertainties. Design decisions are, therefore, based on fuzzy information that is vague, imprecise, qualitative, linguistic, or incomplete. The uncertain information is usually available as intervals with lower and upper limits. In this work, the possibility theory is used to assess design reliability with incomplete information. The possibility theory can be viewed as a variant of fuzzy set theory. The formal theories to handle uncertainty are first introduced using the theoretical fundamentals of fuzzy measures. A computationally efficient and accurate hybrid (global–local) optimization approach is subsequently described for calculating the confidence level of fuzzy response. The method combines the advantages of the commonly used vertex and discretization methods. Subsequently, the possibility theory is used in design. A possibility-based design optimization method is proposed where all design constraints are expressed in a possibilistic way. It is shown that the method gives a conservative solution compared with all conventional reliability-based designs obtained with different probability distributions. Finally, a general possibility-based design optimization method, which handles a combination of random and possibilistic design variables, is presented. Several numerical examples demonstrate the application of possibility theory in design.

I. Introduction

DETERMINISTIC design assumes that there is no uncertainty in the design variables and/or modeling parameters, and therefore there is no variability in the simulation outputs. However, there exists inherent input and parameter variation that results in output variation. Deterministic optimization typically yields optimal designs that are pushed to the limits of design constraint boundaries, leaving little or no room for tolerances (uncertainty) in manufacturing imperfections, modeling and design variables. Therefore, deterministic optimal designs that are obtained without taking into account uncertainty are usually unreliable. Input variation is fully accounted for in reliability-based design optimization (RBDO). Probability distributions describe the stochastic nature of the design variables and model parameters. Variations are represented by standard deviations (typically assumed constant), and a mean performance measure is optimized subject to probabilistic constraints. RBDO has been extensively studied[1−5] because it provides optimum designs in the presence of uncertainty.

In general, classical probabilistic analysis is very effective in design when sufficient data are available to quantify uncertainty using probability distributions. However, when sufficient data are not available or there is a lack of information caused by ignorance, the classical probability methodology might not be appropriate. For example, during the early stages of product development, quantification of the product's reliability or compliance to performance targets is practically very difficult because of insufficient data for modeling the uncertainties. A similar problem exists when the reliability of a complex system is assessed in the presence of incomplete information on the variability of certain design variables, parameters, operating conditions, boundary conditions, etc.

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If a probability distribution is not available because of limited information, there is a tendency to assume a uniform or any other distribution in order to perform an RBDO study. However, this is incorrect because we should not use information that is not known. Lower and upper bounds (intervals) on uncertain design variables are usually known. In this case, interval analysis[6] and fuzzy set theory[7] have been extensively used to characterize input uncertainty and propagate it through a transfer function for calculating the interval of the uncertain output.

An efficient method for reliability estimation with a combination of random and interval variables is presented in Ref. 8. However, it is not implemented in a design optimization framework. There are only a few design optimization studies with some or all of the uncertain design variables being in interval form.[9−12]

In interval analysis, an interval (or range) represents the uncertainty of each input variable. The interval of the uncertain output is then defined by the minimum and maximum of all input endpoint combinations. This has two major drawbacks. First, the computational cost is prohibitively large for a large number of inputs, and second, the output range is only correct for monotonic functions. If the response exhibits local extrema, the interval analysis is usually wrong. An optimal design of mechanical systems with interval uncertain variables is presented in Ref. 10. Because of the inaccuracies of interval analysis calculations, the optimal design can be wrong, unless a physical understanding of the system is known a priori in order to "assist" in correcting the interval calculations.

Optimization with input ranges has also been studied under the term antioptimization.[13,14] Antioptimization is used to describe the task of finding the worst-case scenario for a given problem. It solves a two-level (usually nested) optimization problem. The outer level performs the design optimization while the inner level performs the antioptimization. The latter seeks the worst condition under the interval uncertainty.[14] A decoupled approach is suggested in Ref. 14, where the design optimization alternates with the antioptimization rather than nesting the two. It was mentioned that this method takes longer to converge and might not even converge at all if there is strong coupling between the interval design variables and the rest of the design variables. A worst-case scenario approach with interval variables has also been considered in multidisciplinary systems design.[11,12]

In this paper the possibility theory is used to assess reliability and perform a possibility-based design optimization (PBDO). The possibility theory is viewed as a variant of fuzzy set theory.

A unified approach, which addresses the inaccuracies of interval analysis and antioptimization method based on an efficient hybrid (global–local) optimization, is presented. Deterministic optimization and worst-case antioptimization are special cases of the proposed PBDO method. It is shown that the possibility-based design encompasses all RBDO designs obtained with different input distributions. The method is general, handling problems with either interval inputs only or a combination of random and interval inputs.

The paper is organized as follows. Section II introduces the different types of uncertainty and the uncertainty theories. Because all uncertainty theories are based on the mathematical foundation of fuzzy measures, the fundamentals of fuzzy measures are given in Sec. III. Section IV describes the fundamentals of possibility theory based on fuzzy measures as well as some numerical methods for propagating nonprobabilistic uncertainty, which are essential in possibility-based design. Section V presents a detailed formulation of possibility-based design optimization where design constraints are satisfied possibilistically. All principles are demonstrated with examples. Section VI presents a methodology for design optimization with a combination of random and fuzzy (possibilistic) design variables and compares results with the traditional reliability-based design optimization. Finally, Sec. VII presents the summary and conclusions.

II. Uncertainty Classification and Uncertainty Theories

Uncertainties can be classified in two general types: aleatory (stochastic or random) and epistemic (subjective).[15–19] Aleatory uncertainty results from randomness or unpredictability caused by stochasticity. It is also known as variability or stochastic uncertainty. Aleatory uncertainty is irreducible because it cannot be reduced by collection of more information or data. The classical probability theory models aleatory uncertainty very efficiently. However, when data are scarce or there is lack of information, the probability theory is not useful because the needed probability distributions cannot be accurately constructed. Epistemic uncertainty describes subjectivity, ignorance, or lack of information in any phase of the modeling process. It is reducible because it can be reduced with an increased state of knowledge or collection of more data.

Evidence theory (or Dempster–Shafer theory)[18,19] can be used to handle uncertainty. It includes the possibility[20] and probability[21] theories. Two large classes of fuzzy measures, called belief measures and plausibility measures, characterize the mathematical theory of evidence. They are mutually dual in the sense that one of them can be uniquely determined from the other. Evidence theory uses plausibility and belief (lower bound of probability) to measure the likelihood of events. The belief and plausibility measures are represented by a single function, called basic probability assignment, which assigns degrees of evidence or belief to certain specific propositions or subsets of the universal set. Subsets that are assigned nonzero degrees of evidence are called focal elements. When focal elements are nested, meaning that there is no conflicting evidence or information,[18,22] we obtain a special subclass of dual plausibility and belief measures, called possibility and necessity measures, respectively. Therefore, if there are insufficient data but no conflicting evidence, the possibility theory can be used. An example of conflicting evidence is when two "experts" can assign different chances for the same event to happen.

The dual measures of possibility and necessity are uniquely characterized by the possibility distribution function and the necessity distribution function, respectively, which are analogs to the probability distribution function but with very different normalization requirements. Although in general the largest value of the possibility distribution function is equal to one, its area is not required to be equal to one. However, the area under the probability distribution function is required to be equal to one. When the plausibility and belief measures are equal, the general evidence theory reduces to the classical probability theory.

In summary, the classical probability theory and the possibility theory are subsets of the evidence theory (Fig. 1). Also, there is no overlap between the probability and possibility theories. This section and Sec. III give only a brief but complete summary of the theoretical aspects of fuzzy measures, which provide the basis of uncertainty theories. The interested reader can refer to Refs. 18 and 22 for more detailed information and a variety of examples.

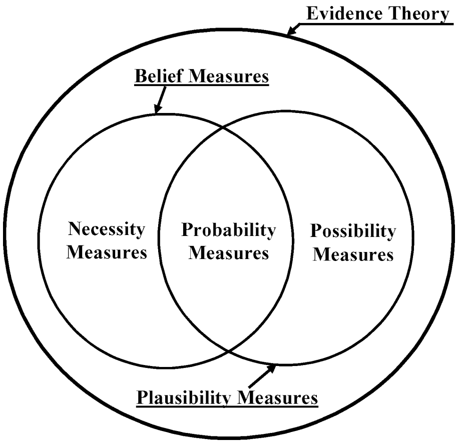

Fig. 1 Fuzzy measures for uncertainty classification.

The probability theory is an ideal tool for formalizing uncertainty when sufficient information is available, or, equivalently, evidence is based on a sufficiently large body of independent random experiments. When there is insufficient information, the possibility theory can be used if there is no conflicting evidence. If there is conflicting evidence, the evidence theory should be used instead. It is very common in real-life problems to have conflicting evidence, even among experts. Finally, when evidence theory is used, the belief and plausibility measures can be interpreted as lower and upper probability estimates, respectively.

Figure 1 shows a pictorial representation of uncertainty classification based on fuzzy measures indicating the application domain and overlaps (or lack thereof) of all available uncertainty theories. This paper concentrates on the fundamentals of possibility theory and how possibility theory can be used in design. Before the fundamentals of possibility theory are described however, an introduction to fuzzy measures is given in the next section. Detailed information is provided in Refs. 17, 18, 20, and 22. The role of fuzzy measures and the axiomatic definition of evidence, possibility, and probability theories are explained.

III. Fuzzy Measures

The description of fuzzy measures provides the foundation of fuzzy set theory. Before we introduce the basics of fuzzy measures, it is helpful to review the used notation on set representation. A universe X represents the entire collection of elements having the same characteristics. The individual elements in the universe X are denoted by x, which are usually called singletons. A set A is a collection of some elements of X. All possible sets of X constitute a special set called the power set $\wp(X)$.

A fuzzy measure is defined by a function $g\colon \wp(X) \to [0, 1]$, which assigns to each crisp[22] subset of X a number in the unit interval [0, 1]. The assigned number in the unit interval for a subset $A \in \wp(X)$, denoted by $g(A)$, represents the degree of available evidence or belief that a given element of X belongs to the subset A.

To qualify as a fuzzy measure, the function g must obey the following three axioms:

Axiom 1 (boundary conditions): $g(\emptyset) = 0$ and $g(X) = 1$.

Axiom 2 (monotonicity): For every $A, B \in \wp(X)$, if $A \subseteq B$, then $g(A) \leq g(B)$.

Axiom 3 (continuity): For every sequence $[A_i \in \wp(X), i = 1, 2, \ldots]$ of subsets of $\wp(X)$, if either $A_1 \subseteq A_2 \subseteq \ldots$ or $A_1 \supseteq A_2 \supseteq \ldots$ (i.e., the sequence is monotonic), then $\lim_{i\to\infty} g(A_i) = g(\lim_{i\to\infty} A_i)$.

In the next subsection, two important and well-developed special types of fuzzy measures, which are usually referred to as belief and plausibility measures, are introduced. These two measures are complementary in the sense that one of them can be uniquely derived from the other.

A. Belief and Plausibility Measures

A belief measure is a function Bel: $\wp(X) \rightarrow [0, 1]$, which satisfies the three axioms of fuzzy measures and the following additional axiom[18]:

$$\mathrm{Bel}(A_1 \cup A_2) \geq \mathrm{Bel}(A_1) + \mathrm{Bel}(A_2) - \mathrm{Bel}(A_1 \cap A_2) \quad (1)$$

Axiom (1) can be expanded for more than two sets. For $A \in \wp(X)$, $\mathrm{Bel}(A)$ is interpreted as the degree of belief, based on available evidence, that a given element of X belongs to the set A. Based on Eq. (1), the following fundamental property of belief measures can be derived:

$$\mathrm{Bel}(A) + \mathrm{Bel}(\bar{A}) \leq 1 \quad (2)$$

Associated with each belief measure is a plausibility measure Pl, which is defined as

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\bar{A}) \quad (3)$$

The following inequality holds for the plausibility measures:

$$\mathrm{Pl}(A) + \mathrm{Pl}(\bar{A}) \geq 1 \quad (4)$$

Every belief measure and its dual plausibility measure can be expressed with respect to the nonnegative function

$$m : \wp(X) \Rightarrow [0, 1] \quad (5)$$

such that $m(\emptyset) = 0$ and

$$\sum_{A \in \wp(X)} m(A) = 1 \quad (6)$$

The function m is called basic probability assignment (BPA) because of the resemblance of Eq. (6) with a similar equation for probability distributions. The basic probability assignment $m(A)$ is interpreted either as the degree of evidence supporting the claim that a specific element of X belongs to the set A or as the degree to which we believe that such a claim is warranted. At this point, it should be noted that the BPA is very different from the probability distribution function. Basic probability assignments are defined on sets of the power set [i.e., on $A \in \wp(X)$], whereas the probability distribution functions are defined on the singletons x of the power set [i.e., on $x \in \wp(X)$]. Every set $A \in \wp(X)$ for which $m(A) > 0$ is called a focal element of m. Focal elements are subsets of X on which the available evidence focuses, that is, available evidence exists.

Given a BPA m, a belief measure and a plausibility measure are uniquely determined by

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B) \quad (7)$$

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq 0} m(B) \quad (8)$$

which are applicable for all $A \in \wp(X)$.

In Eq. (7), $\mathrm{Bel}(A)$ represents the total evidence or belief that the element belongs to A as well as to various subsets of A. The $\mathrm{Pl}(A)$ in Eq. (8) represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets but also the additional evidence or belief associated with sets that overlap with A. Therefore,

$$\mathrm{Pl}(A) \geq \mathrm{Bel}(A) \quad (9)$$

A visual schematic of the belief and plausibility measures is given in Fig. 2.

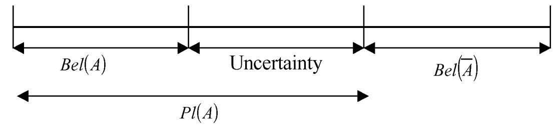

Fig. 2 Belief and plausibility measures.

B. Probability Measures

When the additional axiom of belief measures [see Eq. (1)] is replaced with the stronger axiom

$$\mathrm{Bel}(A \cup B) = \mathrm{Bel}(A) + \mathrm{Bel}(B) \qquad \text{where} \qquad A \cap B = \emptyset \quad (10)$$

we obtain a special type of belief measures, which are the classical probability measures. In this case, the right-hand sides of Eqs. (7) and (8) become equal, and therefore

$$\mathrm{Bel}(A) = \mathrm{Pl}(A) = \sum_{x \in A} m(x) = \sum_{x \in A} p(x) \quad (11)$$

for all $A \in \wp(X)$, where $p(x)$ is the classical probability distribution function (PDF). Note that the BPA $m(x)$ is equal to $p(x)$.Therefore with evidence theory, we can simultaneously handle a mixture of input parameters. Some of the inputs can be described probabilistically (random uncertainty), and some can be described through expert opinions (epistemic uncertainty with incomplete data). In the second case, the range of each input parameter will be discretized using a finite number of intervals. The BPA value for each interval must be equal to the PDF area within the interval.

According to evidence theory, the $\mathrm{Bel}(A)$ and $\mathrm{Pl}(A)$ bracket the true probability $P(A)$ (Refs. 18 and 19), that is,

$$\mathrm{Bel}(A) \leq P(A) \leq \mathrm{Pl}(A) \quad (12)$$

C. Possibility and Necessity Measures

The possibility theory is a subcase of the general evidence theory. It can be used to characterize epistemic uncertainty, when incomplete data are available. It applies only when there is no conflict in the provided body of evidence. In such a case, the focal elements of the body of evidence are nested, and the associated belief and plausibility measures are called consonant. In contrary, when there is conflicting evidence, the belief and plausibility measures are dissonant. A family of subsets of the universal set is nested if they can be ordered in such a way that each is contained within the next. Thus, $A_1 \subset A_2 \subset \cdots \subset A_n$ are nested sets. When the focal elements are nested, the following relations characterize the belief and plausibility measures[17,18]:

$$\mathrm{Bel}(A \cap B) = \min[\mathrm{Bel}(A), \mathrm{Bel}(B)] \qquad \text{for all} \qquad A, B \in \wp(X) \quad (13a)$$

$$\mathrm{Pl}(A \cup B) = \max[\mathrm{Pl}(A), \mathrm{Pl}(B)] \qquad \text{for all} \qquad A, B \in \wp(X) \quad (13b)$$

Consonant belief and plausibility measures are usually known as necessity measures and possibility measures, respectively. If n and π denote a necessity and possibility measure respectively, Eqs. (13a) and (13b) can be expressed as

$$n(A \cap B) = \min[n(A), n(B)] \quad (14)$$

$$\pi(A \cup B) = \max[\pi(A), \pi(B)] \quad (15)$$

Necessity and possibility measures do not satisfy the additivity axiom. In fact, we can easily verify that

$$\max[\pi(A), \pi(\bar{A})] = 1, \qquad \pi(A) + \pi(\bar{A}) \geq 1 \quad (16a)$$

$$\min[n(A), n(\bar{A})] = 0, \qquad n(A) + n(\bar{A}) \leq 1 \quad (16b)$$

The necessity and possibility are dual measures, related by

$$n(A) = 1 - \pi(\bar{A}) \quad (17)$$

If one of the necessity and possibility measures is known, the other can be calculated from Eq. (17).

IV. Fundamentals of Possibility Theory

This section highlights the fundamentals of possibility theory as it was originally introduced in the context of fuzzy set theory.[23] In the fuzzy set approach to possibility theory, focal elements are represented by a cuts of the associated fuzzy set. Focal elements are subsets that are assigned nonzero degrees of evidence. The possibility theory can be used to bracket the true probability based on the fuzzy set approach at various confidence intervals (a cuts). The advantage of this is that as the design progresses and the confidence level on the input parameter bounds increases, the design need not be reevaluated to obtain the new bounds of the response. The possibility theory is a subcase of the general evidence theory when there is no conflicting evidence in characterizing epistemic uncertainty.

Similarly to the probability measures, which are represented by the probability distribution functions, the possibility measures can be represented by the possibility distribution function $r : X \Rightarrow [0, 1]$ such that

$$\pi(A) = \max_{x \in A} (x) \tag{18}$$

It can be shown that possibility measures are formally equivalent to fuzzy sets. In this equivalence, the membership grade of an element x corresponds to the plausibility of the singleton consisting of that x. Therefore, a consonant belief structure is equivalent to a fuzzy set of X.

A fuzzy set is an imprecisely defined set that does not have a crisp boundary. It provides instead a gradual transition from "belonging" to "not belonging" to the set. A function can be defined such that the values assigned to the elements of the set are within a specified range and indicate the membership grade of these elements in the set. Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set.

The membership function μ_A by which a fuzzy set A is usually defined has the form $\mu_A : X \rightarrow [0, 1]$, where [0, 1] denotes the interval of real numbers from 0 to 1, inclusive. Given a fuzzy subset A of X with membership function μ_A, Zadeh[23] defines a possibility distribution function r associated with A as numerically equal to μ_A, that is, $r(x) = \mu_A(x)$ for all $x \in X$. Then, he defines the corresponding possibility measure π as

$$\pi(A) = \sup_{x \in A} r(x) \qquad \text{for each} \qquad A \in \wp(X) \tag{19}$$

Equation (19) is equivalent to Eq. (18) when X is finite. In the fuzzy set approach to possibility theory, focal elements are represented by a cuts of the associated fuzzy set. For the remainder of this discussion, we will follow the fuzzy set approach to possibility theory.

In Sec. III.B, we described a methodology to bracket the true probability with the belief and plausibility measures. If we know the possibility distribution function $\mu_Y(y)$ of the response Y, then the true probability $P(Y)$ can be also bracketed as

$$n(Y) \leq P(Y) \leq \pi(Y) \tag{20}$$

where the necessity $n(Y)$ and possibility $\pi(Y)$ measures are calculated from Eqs. (19) and (17), respectively. The extension principle[18,19,22] is used to calculate the possibility distribution function $\mu_Y(y)$ of the response.

A. Fuzzification Process and Extension Principle

The process of quantifying a fuzzy variable is known as fuzzification. If any one of the input variables is imprecise, it is considered fuzzy and must therefore be fuzzified for the uncertainty to be propagated using fuzzy calculus. The fuzzification is done by constructing a possibility distribution, or membership function, for each imprecise (fuzzy) variable. Details can be found in Ref. 22. The membership function takes values in the [0, 1] interval. Here, we use convex normal possibility distributions to characterize the fuzzy variables. An example of a convex normal triangular possibility distribution is shown in Fig. 3. The point for which the possibility is equal to one is called normal point. The possibility distribution is convex because it is strictly decreasing to the left and right of the normal point. At each confidence level, or a cut, a set X_a is defined as

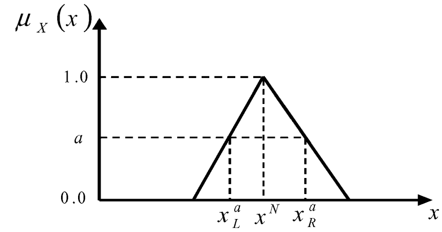

Fig. 3 Triangular possibility distribution for a fuzzy variable.

$$X_a = \left\{x : x_L^a \leq x \leq x_R^a, a \in [0, 1]\right\} \tag{21a}$$

which is a monotonically decreasing function of a, that is,

$$a_1 > a_2 \Rightarrow X_{a_1} \subset X_{a_2} \qquad \text{for every} \qquad a_1, a_2 \in [0, 1] \tag{21b}$$

Because of the convexity of the possibility distribution function, all sets generated at different a cuts are nested according to Eq. (21b). Therefore, the convexity and normality of the possibility distribution function satisfies the basic requirement of nested sets (no conflicting evidence) in possibility theory.

After the fuzzification of the imprecise input variables, the extension principle is used to propagate the epistemic uncertainty through the transfer function in order to calculate the fuzzy response. The extension principle calculates the possibility distribution of the fuzzy response from the possibility distributions of the fuzzy input variables. In particular, given the transfer function $y = f(x)$, where the output y depends on the N independent fuzzy inputs $x = \{x_1, \ldots, x_N\}$, the extension principle states that the possibility distribution μ_Y of the output is given by

$$\mu_Y[y = f(\underset{\sim}{x})] = \sup_y \left\{\min_j \lfloor \mu_X[f \underset{\sim}{x}_j)] \rfloor\right\} \tag{22}$$

where sup denotes the suprenum operator that gives the least upper bound. The preceding equation can be interpreted as follows. For a crisp value of the output y, there can exist more than one combination of crisp values of input variables $\underset{\sim}{x}$ resulting in the same output. The possibility of each combination is given by the smallest possibility value for all fuzzy input variables. The possibility that $y = f(\underset{\sim}{x})$ is given by the maximum possibility for all of these combinations. Note that in probability theory the probability of an outcome is equal to the product of the probabilities of the constituent events. In fuzzy set theory, however, the possibility of an outcome is equal to the minimum possibility of the constituent events. If the outcome can be reached in many ways, then the outcome probability, in probability theory, is given by the sum of the probabilities of all of the ways. In fuzzy theory, the possibility of the outcome is given by the maximum possibility of all of the possibilities.[22]

The direct ("brute force") solution of Eq. (22) is practically intractable except for simple cases involving one or two fuzzy variables. The computational effort increases exponentially with increasing number of fuzzy input variables. For this reason, approximate numerical techniques have been proposed, among which the discretization method[24] and the vertex method[25,26] are the most popular ones.

In the discretization method, the domain of each fuzzy variable i; $1 \leq i \leq N$ is discretized with M_i discrete values at each a cut. Then the output y is evaluated at all possible combinations

$$\prod_{i=1}^{N} M_i$$

for each a cut. Subsequently, Eq. (22) is used to calculate the possibility distribution of the output. The range of the output is defined by the minimum and maximum response from all combinations. Although this method can be very accurate, the associated computational cost is practically prohibitive.

In the vertex method, all of the binary combinations of only the extreme values of the fuzzy variables at an a cut are fed into the deterministic transfer function. The bounds of the fuzzy response are then obtained at the a cut, by choosing the maximum and minimum responses. The procedure is repeated for all a cuts of interest.[26] The method has the potential to give accurate bounds of the response based on the bounded input. However, when the transfer function exhibits minima or maxima within the domain defined by the extreme values of the input variables, the vertex method is inaccurate. This is because the function is evaluated only at the binary combinations of the input variable bounds. For a problem with N fuzzy input variables, the required number of function evaluations for the vertex method is $A * 2^N$, where A is the number of a cuts.

In general, the vertex method is computationally more efficient compared with the discretization method. However, the required computational effort grows exponentially with the number of input fuzzy variables.[22] For this reason, most of the reported applications are restricted to very few fuzzy variables.[27–29]

A hybrid (global–local) optimization method is presented in this paper, which ensures computational efficiency without loss of accuracy. An optimization algorithm is used to calculate the minimum and maximum values of the response at each a cut. Because we need the global minimum and maximum values of the response, a global optimizer is used in order to avoid being trapped at a local optimum and obtain, therefore, an inaccurate solution. The DIRECT (divisions of rectangles) global optimizer is used in this work. DIRECT is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant.[30] Although global optimizers can get close to the global optimum quickly, it takes them longer to achieve a high degree of accuracy because they are nonderivative based and, therefore, have a slow rate of convergence. This suggests that the best performance can be obtained by combining DIRECT with a gradient-based local optimizer in a hybrid approach. In this work, DIRECT is first used, followed by a local optimizer based on sequential quadratic programming (SQP). The number of function evaluations increases until DIRECT provides a converged global optimum based on "loose" convergence criteria. Subsequently, the DIRECT solution is used as starting point for SQP, which identifies the optimum accurately and efficiently.

B. Numerical Examples

In this section, two examples demonstrate the accuracy and efficiency of the hybrid optimization method of Sec. IV.A and compare it with the vertex and discretization methods. A mathematical example is first presented followed by a vehicle side impact example.

1. Mathematical Example

The following two-variable, six-hump camel function[31] is first used:

$$y(x_1, x_2) = 4x_1^2 - 2.1x_1^4 + \tfrac{1}{3}x_1^6 + x_1 x_2 - 4x_2^2 + 4x_2^4$$

$$x_{1,2} \in [-2, 2]$$

For demonstration reasons, the following simple triangular membership functions are used for the two input variables x_1 and x_2:

$$\mu_{X_i}(x_i) = \begin{cases} -x_i/2 + 1, & 0 \le x_i \le 2 \\ x_i/2 + 1, & -2 \le x_i \le 0 \end{cases} \qquad i = 1, 2$$

Figure 4 shows the contour plot of the six hump camel function. The Hs indicate all extreme points. Points H2 and H5 with coordinates (0.0898, −0.7127) and (−0.0898, 0.7127), respectively, are two global optima with an equal function value of $y_{\min} = -1.0316$.

The calculated membership functions of the response y by the vertex, discretization, and hybrid optimization methods are plotted in Fig. 5. Ten a cuts are used for all three methods. For the discretization method, the range of each input fuzzy variable, at each a cut, is equally split in 15 divisions. It is known that if the input membership functions are convex normal, the response membership function must also be convex normal. The justification is that when the input uncertainty increases (low a cut values), the uncertainty of the response must remain the same or increase. As shown in Fig. 5, the response membership function obtained by the vertex method is not convex, and therefore it is wrong.

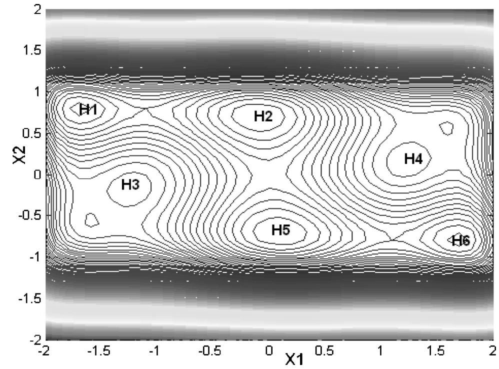

Fig. 4 Contour plot for mathematical example.

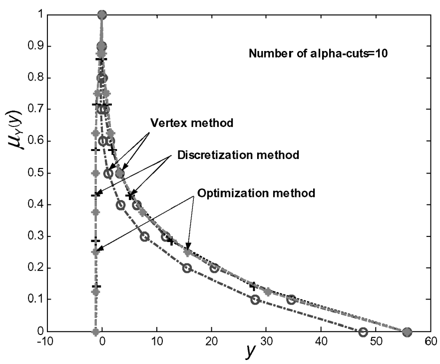

Fig. 5 Membership function of response for mathematical example.

As explained in Sec. IV.A, the discretization method evaluates the function not only at the upper and lower limits of the input variables at each alpha cut but also between the bounds. Thus, it can capture the extreme points that might be present in between the upper and lower bounds. At each alpha cut, all combinations are obtained, and the minimum and maximum response values are calculated to get the response membership function. It is clear that the response becomes more accurate as the number of divisions per alpha cut increases. As shown in Fig. 5, the response membership function calculated with the discretization method is convex normal. The uncertainty decreases as the level of confidence increases (increasing a-cut values). The major disadvantage of this method is that as the number of design variables increases and the number of divisions per a cut also increases, the method becomes computationally very expensive. In this example, the number of a cuts is 10, and the number of divisions per a cut is 15. Therefore, the number of function evaluations is $10 \times (15 + 1)^2 = 2560$.

The response membership function of the six hump camel function is also calculated using the proposed hybrid optimization method. The result is identical with that obtained with the discretization method (see Fig. 5).

Table 1 summarizes the lower and upper bound values of the response at the zero a cut, as calculated by the vertex, discretization, and hybrid optimization methods. The vertex method is very efficient but inaccurate. The hybrid optimization method, however, has the same accuracy with the brute force discretization method, but it is much more efficient.

2. Vehicle Side Impact Example

A vehicle side impact example is used to further demonstrate the accuracy and efficiency of the proposed hybrid optimization method and compare it with the vertex and discretization methods. Uncertainties in structural design variables, material properties, and operating conditions, among others, are very important in automotive vehicle side impact studies. For this reason, vehicle crashworthiness under uncertainty has recently gained considerable attention.[32–34]

The side impact is an optimization problem, which is formed by minimizing the vehicle weight subject to a number of safety constraints on the dummy including head injury criterion, abdomen load, rib deflections (upper, middle and lower), chest viscous criterion, and pubic symphysis force. Eleven design variables are used to describe the side impact problem. Table 2 shows sequentially the description of the design variables, their lower and upper bounds, and nominal value. Because of the excessive computational cost associated with vehicle crashworthiness, a set of response surfaces is usually used. A full description of the problem is provided in Ref. 2.

Simple triangular membership functions are used for all 11 design variables. The lower and upper bound values at zero a cut are equal to the minimum and maximum values respectively, given in Table 2. The normal points are equal to the nominal values. The membership functions of all constraints are calculated using the vertex, discretization, and hybrid optimization methods. As an example, the membership function of the pubic symphysis force (Pubic_F) is shown in Fig. 6. The used response surface for Pubic_F is

$$\text{Pubic\_F} = 4.72 - 0.5x_4 - 0.19x_2x_3 - 0.0122x_4x_{10} + 0.009325x_6x_{10} + 0.000191x_{11}x_{11}$$

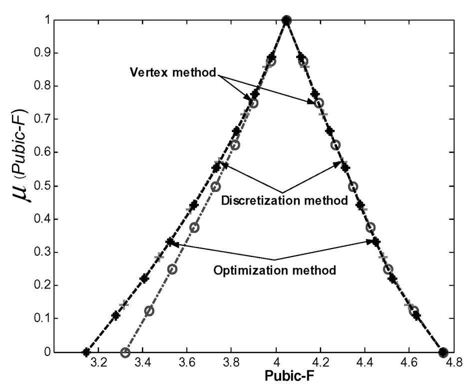

Fig. 6 Calculated membership function for pubic symphysis force.

Although all three methods calculate the upper bound of Pubic_F accurately at each a cut, the vertex method gives inaccurate estimates of the lower bounds.

Table 3 shows the lower and upper bound values of Pubic_F at the zero a cut level and the number of function evaluations for all three methods. The proposed hybrid optimization method has the same accuracy with the discretization method but much better efficiency. Because the hybrid optimization method takes care of the internal extremes, it is accurate and computationally efficient.

V. Possibility-Based Design Optimization

In deterministic design optimization, an objective function is minimized subject to satisfying each constraint. In RBDO, where all design variables are characterized probabilistically, an objective function is usually minimized subject to the probability of satisfying each constraint being greater than a specified high reliability level.

In this section, a methodology is presented on how to use possibility theory in design. We will show that the possibility-based design is conservative compared with all RBDO designs obtained with different probability distributions. In RBDO, some optimality is usually sacrificed in order to accommodate the random uncertainty. The possibility-based design sacrifices a little more optimality in order to accommodate the lack of probability distribution information. It therefore, encompasses all RDBO designs obtained with different distributions.

In Sec. III, we showed that the probability $P(A)$ of event A is bracketed by the belief Bel(A) and plausibility Pl(A), that is, $\text{Bel}(A) \le P(A) \le \text{Pl}(A)$. Note that according to the probability theory, $\text{Bel}(A) = P(A) = \text{Pl}(A)$. We have also mentioned that for consonant (no conflicting evidence) belief structures, the plausibility measures are equal to the possibility measures, resulting in $\eta(A) \le P(A) \le \pi(A)$, where η and π are the necessity and possibility measures, respectively [see Eq. (20)]. This shows that the possibility $\pi(A)$ provides an upper bound to the probability $P(A)$.

From the design point of view, we can therefore conclude[18,22,23] that 1) what is possible might not be probable and 2) what is impossible is also improbable.

Note that for an impossible event A, the possibility $\pi(A)$ is zero. If we, therefore, make sure that the possibility of violating a constraint is zero, then the probability of violating the same constraint will be also zero. If feasibility of a constraint g is expressed with the positive null form $g \ge 0$, the constraint is always satisfied if

$$\pi(g \le 0) = 0 \tag{23}$$

The possibility π in Eq. (23) is calculated using Eq. (19). Figure 7 shows the membership function $\mu_G(g)$ of constraint g. The

Table 1 Accuracy and efficiency comparison of vertex, discretization, and hybrid optimization methods

Parameter	Vertex	Discretization	Hybrid optimization
Lower bound	47.73	−1.01	−1.03
Upper bound	55.73	55.73	55.73
No. of function evaluations	4	256	140

Table 3 Accuracy and efficiency comparison of vertex, discretization, and hybrid optimization methods for function Pubic_F

Parameter	Vertex	Discretization	Hybrid optimization
Lower bound	3.32	3.14	3.14
Upper bound	4.75	4.75	4.74
No. of function evaluations	64	15625	242

Table 2 Description and range of design variables for vehicle side impact example

Number	Design variable description	Range		
		Minimum	Nominal	Maximum
1	Thickness of B-Pillar, inner, mm	0.5	1.0	1.5
2	Thickness of B-Pillar, reinforcement, mm	0.45	0.9	1.35
3	Thickness of floor side, inner, mm	0.5	1.0	1.5
4	Thickness of cross members 1 and 2, mm	0.5	1.0	1.5
5	Thickness of door beam, mm	0.875	1.75	2.625
6	Thickness of door belt line, reinforcement, mm	0.4	0.8	1.2
7	Thickness of roof rail, mm	0.4	0.8	1.2
8	Material of B-Pillar, inner	Mild steel	Mild steel	HS steel
9	Material of floor side, inner	Mild steel	Mild steel	HS steel
10	Barrier height, mm	−30	0	+30
11	Barrier hitting position, mm	−30	0	+30

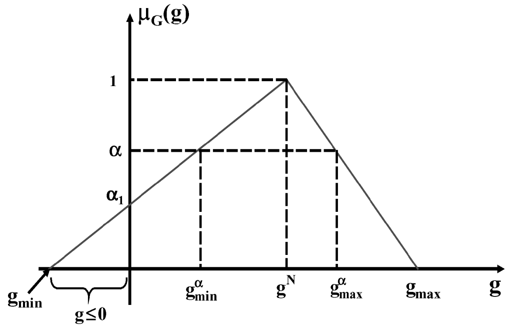

Fig. 7 Used notation in possibility-based design optimization.

possibility of set $A = \{g : g_{\min} \le g \le g^{\alpha}_{\min}, \alpha \in [0,1]\}$ is $\pi(A) = \alpha$ and the possibility of set $B = \{g : g^{\alpha}_{\min} \le g \le g^{\alpha}_{\max}, \alpha \in [0,1]\}$ is $\pi(B) = 1$. Similarly, the possibility of constraint violation is $\pi(g \le 0) = \alpha_1$.

Equation (23) can be relaxed as

$$\pi(g \le 0) \le \alpha \tag{24}$$

where the a-cut level is small, that is, $\alpha \ll 1$. Based on Fig. 7, the relation (24) is satisfied if

$$g^{\alpha}_{\min} \ge 0 \tag{25}$$

where $g^{\alpha}_{\min}$ is the global minimum of g at the a cut. Equation (25) is analogous to the R-percentile formulation [1, 2] of a probabilistic constraint in RBDO. The possibilistic constraint of Eq. (24) or (25) becomes active if $g^{\alpha}_{\max} = 0$.

A. Formulation of PBDO

Based on the preceding discussion, a PBDO problem can be formulated as

$$\begin{aligned} &\min_{d, X^N} f(\boldsymbol{d}, \boldsymbol{X}^N, \boldsymbol{P}) \\ \text{s.t.} \quad &\pi[G_i(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \le 0] \le \alpha, \qquad i = 1, \ldots, n \\ &\boldsymbol{d}_L \le \boldsymbol{d} \le \boldsymbol{d}_U \\ &\boldsymbol{X}_L \le \boldsymbol{X}^N \le \boldsymbol{X}_U \end{aligned} \tag{26}$$

where $\boldsymbol{d} \in R^k$ is the vector of deterministic design variables; $\boldsymbol{X} \in R^m$ is the vector of possibilistic design variables; $\boldsymbol{P} \in R^q$ is the vector of possibilistic design parameters; $f()$ is the objective function; and n, k, m, and q are the number of constraints, deterministic design variables, possibilistic design variables, and possibilistic design parameters, respectively. According to the used notation, a bold letter indicates a vector, an uppercase letter indicates a possibilistic variable or parameter, and a lowercase letter indicates a realization of the possibilistic variable or parameter. Feasibility of the ith deterministic constraint is expressed with the positive null form $G_i \ge 0$.

The possibilistic design variables are represented with convex normal possibility distributions (membership functions) similar to the one shown in Fig. 3. Note that they might not be necessarily triangular. The superscript N denotes the normal point of each distribution, where the membership function value is equal to one. Subscripts L and U denote lower and upper bounds, respectively. In PBDO, we will assume that the membership functions of the possibilistic design variables have a constant shape and that their normal points are design variables moving within predetermined bounds. This is analogous to RBDO in which the PDF of each random design variable stays constant and its mean value is a design variable.

Based on Eq. (25), the PBDO formulation (26) is equivalent to

$$\begin{aligned} &\min_{d, X^N} f(\boldsymbol{d}, \boldsymbol{X}^N, \boldsymbol{P}) \\ \text{s.t.} \quad &g^{\alpha}_{i_{\min}} \ge 0, \qquad i = 1, \ldots, n \\ &\boldsymbol{d}_L \le \boldsymbol{d} \le \boldsymbol{d}_U \\ &\boldsymbol{X}_L \le \boldsymbol{X}^N \le \boldsymbol{X}_U \end{aligned} \tag{27}$$

Note that the PBDO formulation (26) or (27) is a double-loop optimization problem in which an optimization is performed (inner loop) when the design optimization (outer loop) calls for a possibilistic constraint evaluation. In general, a global optimization must be performed in the inner loop. The PBDO optimum at $a = 1$ coincides with the deterministic optimum.

B. Examples

In this section, the possibility-based design optimization is demonstrated with a mathematical example and a cantilever beam example. Both examples have been used in detailed RBDO studies in Ref. 2. They are chosen here so that the possibility-based design results are compared with the existing reliability-based design results. Theoretically, the possibility and reliability-based results cannot be compared because the possibility and reliability theories are different from each other. However, for practical purposes, we attempt to compare them by arbitrarily using membership functions that "resemble" the probability density functions used in the reliability-based studies.

1. Mathematical Example

This example has two possibilistic variables x_1, x_2 and three nonlinear constraints g_1, g_2, g_3. The objective function is simply the sum of the normal values of the two possibilistic variables. The PBDO problem is stated as

$$\begin{aligned} &\min_{x_1^N, x_2^N} f = x_1^N + x_2^N \\ \text{s.t.} \quad &g^{\alpha}_{j_{\min}} \ge 0, \qquad j = 1 \sim 3 \\ &0 \le x_i^N \le 10, \qquad i = 1 \sim 2 \\ &g_1(\boldsymbol{X}) = x_1^2 * x_2/20 - 1 \\ &g_2(\boldsymbol{X}) = (x_1 + x_2 - 5)^2/30 + (x_1 - x_2 - 12)^2/120 - 1 \\ &g_3(\boldsymbol{X}) = 80\big/\left(x_1^2 + 8 * x_2 + 5\right) - 1 \end{aligned}$$

For demonstration prurposes, each possibilistic design variable x_i is arbitrarily represented with a triangular membership function $(x_i^N - 3 * 0.3, x_i^N, x_i^N + 3 * 0.3)$, where x_i^N is the normal point and $(x_i^N - 3 * 0.3, x_i^N + 3 * 0.3)$ is the range of x_i at zero a cut. In the RBDO study of Ref. 2, the design variables x_1, x_2 are normally distributed with standard deviations equal to 0.3. The chosen ranges extend three standard deviations from each side of the normal point in an attempt to use a similar variation with the RBDO study.

Table 4 summarizes the results of the possibility-based design optimization and compares them with the deterministic optimum and the RBDO results. The second and third columns show the deterministic optimization and RBDO results.[2] A reliability index $\beta = 3$ has been used in the RBDO study for all constraints. The last four columns show the PBDO results for the 0.4, 0.2, 0.1, and 0 a cuts. As expected, the deterministic optimum is smaller than the RBDO optimum, which in turn is smaller than the PBDO optimum at the zero a cut. The PBDO optimum of 7.1804 at $a = 0$ provides an upper bound. For higher a cuts, the PBDO optimum decreases becoming even less than the RBDO optimum at $a = 0.4$. For all cases, the first two constraints are active, and the third constraint is inactive.

2. Cantilever Beam Example

In this example, a cantilever beam in vertical and lateral bending[3] is used (Fig. 8). The beam is loaded at its tip by the vertical and

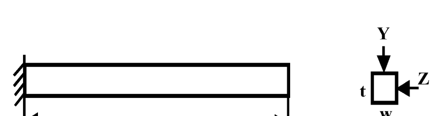

Fig. 8 Cantilever beam under vertical and lateral bending.

Table 4 Comparison of PBDO and RBDO optima for the mathematical example

Design variable	Deterministic optimum	Reliability optimum	Possibility-based optimum			
			$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0$
x_1^N	3.1139	3.4391	3.3702	3.508	3.5844	3.6657
x_2^N	2.0626	3.2866	3.0369	3.293	3.410	3.5147
Objective						
$f(\boldsymbol{X})$	5.1765	6.7257	6.4071	6.8011	6.9927	7.1804
Constraints						
$g_1(\boldsymbol{X})$	0	0	0	0	0	0
$g_2(\boldsymbol{X})$	0	0	0	0	0	0
$g_3(\boldsymbol{X})$	1.564	0.50	0.635	0.455	0.378	0.308

Table 5 Comparison of PBDO and RBDO optima for the cantilever beam example

Design variable	Deterministic optimum	Reliability optimum	Possibility-based optimum			
			$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0$
w	2.0470	2.4781	2.5424	2.4692	2.5298	2.5901
t	3.7459	3.8421	3.7658	4.1376	4.1726	4.210
Objective						
$f(\boldsymbol{X})$	7.6679	9.5212	9.574	10.217	10.556	10.901
Constraints						
$g_1(\boldsymbol{X})$	0	0.0038	0	0	0	0
$g_2(\boldsymbol{X})$	0	0.359	0.408	0.33	0.375	0.420

lateral loads Y and Z, respectively. Its length L is equal to 100 in. (1 in. = 25.4 mm). The width w and thickness t of the cross section are deterministic design variables. The objective is to minimize the weight of the beam. This is equivalent to minimizing $f = w * t$, assuming that the material density and the beam length are constant.

Two nonlinear failure modes are used. The first failure mode is yielding at the fixed end of the cantilever; the other failure mode is that the tip displacement exceeds the allowable value $D_0 = 2.5$ in. The PBDO problem is formulated as

$$\min_{w,t} \ f = w * t$$

$$\text{s.t.} \qquad g_{i_{\min}}^{\alpha} \geq 0, \qquad j = 1 \sim 2$$

$$g_1(y, Z, Y, w, t) = y - \left(\frac{600}{wt^2} * Y + \frac{600}{w^2 t} * Z\right)$$

$$g_2(E, Z, Y, w, t) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{Z}{w^2}\right)^2}$$

$$0 \leq w, t \leq 5$$

where g_1 and g_2 are the limit states corresponding to the two failure modes. The design variables w and t are deterministic. In the RBDO study of Ref. 2, Y, Z, y, and E are normally distributed random parameters with $Y \sim N$ (1000, 100) lb (1 lb = 0.45359 kg), $Z \sim N$ (500, 100) lb, $y \sim N$ (40,000, 2000) psi (1psi = 6.8948 kPa), and $E \sim N$ (29×10^6, 1.45×10^6) psi; y is the random yield strength, Z and Y are mutually independent random loads in the vertical and lateral directions, respectively, and E is the Young modulus. A reliability index $\beta = 3$ has been used for both constraints.

In the current PBDO study, Y, Z, y, and E are possibilistic parameters described with the triangular membership functions $(x^N - 3 * \sigma, x^N, x^N + 3 * \sigma)$, where x^N is the normal point of each variable and σ is the used standard deviation in the RBDO study. Table 5 compares the deterministic optimization, RBDO, and PBDO results. Similar conclusions with the preceding mathematical example can be drawn. The PBDO optimum with $a = 0$ is higher that the RBDO optimum and provides an upper bound of all RBDO optima obtained with different distributions, as long as these distributions have similar variability ranges (e.g., different beta distributions defined over the same range). For higher a cut values the PBDO optimum reduces. The PBDO optimum at $a = 1$ coincides with the deterministic optimum. In this example, although both constraints are active at the deterministic optimum, only the first constraint is active for both the RBDO and PBDO optima.

VI. PBDO with a Combination of Random and Possibilistic Variables

Deterministic optimization typically yields optimal designs that are pushed to the limits of design constraint boundaries, leaving little or no room for uncertainty in manufacturing imperfections, modeling, and design variables. Deterministic optimal designs that are obtained without taking into account uncertainty are therefore usually unreliable.

RBDO provides optimum designs in the presence of only *random* (or aleatory) uncertainty.[1–3] A typical RBDO problem is formulated as[2]

$$\min_{d,\mu_X} \ f(\boldsymbol{d}, \boldsymbol{\mu}_X, \boldsymbol{\mu}_P)$$

$$\text{s.t.} \qquad P[G_i(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \geq 0] \geq R_i = 1 - p_{f_i}, \qquad i = 1, \ldots, n$$

$$\boldsymbol{d}^L \leq \boldsymbol{d} \leq \boldsymbol{d}^U$$

$$\boldsymbol{\mu}_X^L \leq \boldsymbol{\mu}_X \leq \boldsymbol{\mu}_X^U \tag{28}$$

where $\boldsymbol{d} \in R^k$ is the vector of deterministic design variables, $\boldsymbol{X} \in R^m$ is the vector of random design variables, and $\boldsymbol{P} \in R^q$ is the vector of random design.

For a variety of practical applications, however, there might not be enough information to characterize all design variables probabilistically. It is, therefore, possible to characterize a subset of the design variables possibilistically by using membership functions according to the methodology presented in Secs. IV and V. In this section, the classical RBDO theory is extended so that it considers a combination of random and possibilistic (fuzzy) design variables, where the former are characterized probabilistically and the latter possibilistically.

A. Formulation

A PBDO problem with a combination of random and possibilistic (or fuzzy) variables can be formulated as

$$\min_{d,X^N,\mu_Y} \ f(\boldsymbol{d}, \boldsymbol{\mu}_Y, \boldsymbol{\mu}_Z, \boldsymbol{X}^N, \boldsymbol{P})$$

$$\text{s.t.} \qquad g_{i_{\min}}^{\alpha} \geq 0, \qquad i = 1, \ldots, n$$

$$\boldsymbol{d}_L \leq \boldsymbol{d} \leq \boldsymbol{d}_U$$

$$\boldsymbol{\mu}_Y^L \leq \boldsymbol{\mu}_Y \leq \boldsymbol{\mu}_Y^U$$

$$\boldsymbol{X}_L \leq \boldsymbol{X}^N \leq \boldsymbol{X}_U$$

Table 6 Comparison of PBDO and RBDO optima for the mathematical example; one possibilistic and one random variable

	Design variable		Objective	Constraint		
Parameter	x_1^N	μ_2	$f(X)$	$g_1(X)$	$g_2(X)$	$g_3(X)$
Deterministic optimum	3.1139	2.0626	5.1765	0.00	0.00	1.5643
Reliability optimum ($\beta = 3$)	3.4391	3.2866	6.7257	0.00	0.00	0.50
Possibility-based optimum						
$\alpha = 0.4$, $\beta \geq 3$	3.3702	3.3969	6.7671	0.00	0.00	10.5563
$\alpha = 0.4$, $\beta \geq 3.2$	3.3702	3.4569	6.8271	0.00	0.00	10.1563
$\alpha = 0.4$, $\beta \geq 3.3$	3.3702	3.4869	6.8571	0.00	0.00	9.9563
$\alpha = 0.2$, $\beta \geq 3$	3.5079	3.4732	6.9811	0.00	0.00	9.2250
$\alpha = 0.2$, $\beta \geq 3.2$	3.5079	3.5332	7.0411	0.00	0.00	8.8248
$\alpha = 0.2$, $\beta \geq 3.3$	3.5079	3.5632	7.0711	0.00	0.00	8.6248
$\alpha = 0.1$, $\beta \geq 3$	3.5842	3.4983	7.0825	0.00	0.00	8.5440
$\alpha = 0.1$, $\beta \geq 3.2$	3.5842	3.5583	7.1425	0.00	0.00	8.1436
$\alpha = 0.1$, $\beta \geq 3.3$	3.5842	3.5883	7.1725	0.00	0.00	7.9436
$\alpha = 0$, $\beta \geq 3$	3.6657	3.5147	7.1804	0.00	0.00	7.8487
$\alpha = 0$, $\beta \geq 3.2$	3.6657	3.5747	7.2404	0.00	0.00	7.4487
$\alpha = 0$, $\beta \geq 3.3$	3.6657	3.6047	7.2704	0.00	0.00	7.2487

with

$$g_{i_{\min}}^{\alpha} = \min_{X} \left(\beta_i - \beta_{t_i}\right) \geq 0, \qquad i = 1, \ldots, n$$

$$X_L^{\alpha} \leq X \leq X_U^{\alpha}$$

$$\beta = \min_{U} \|U\|$$

$$\text{s.t.} \qquad G(U) = 0 \tag{29}$$

where $d \in R^k$ is the vector of deterministic design variables, $X \in R^m$ is the vector of possibilistic design variables, $P \in R^q$ is the vector of possibilistic design parameters, $Y \in R^\ell$ is the vector of random design variables, $Z \in R^r$ is the vector of random design parameters, and β_t is the target reliability index.

All constraints in problem (29) are expressed possibilistically because the probability theory is a subset of the possibility theory. The described formulation represents a triple-loop optimization sequence. The design optimization of the outer loop calls a series of possibilistic constraints in the middle loop. Each possibilistic constraint is, in general, a global optimization problem. Finally, each possibilistic constraint is a function of the corresponding reliability index β, which represents the third loop of the optimization sequence. For computational purposes, two of the three nested loops can be easily combined.

B. Example

The mathematical example of Sec. V.B.1 is used here as well. The design variable x_1 is possibilistic, and the design variable x_2 is normally distributed with $N(\mu_2, 0.3)$. The triangular membership function $(x_1^N - 3*0.3, x_1^N, x_1^N + 3*0.3)$ is used for x_1. The PBDO problem is stated as

$$\min_{x_1^N, \mu_2} f = x_1^N + \mu_2$$

$$0 \leq x_1^N, \mu_2 \leq 10$$

$$\text{s.t.} \qquad g_{i_{\min}}^{\alpha} \geq 0, \qquad i = 1 \sim 3$$

where

$$g_{i_{\min}}^{\alpha} = \min_{x_1} \left(\beta_i - \beta_{t_i}\right)$$

$$x_{1L}^{\alpha} \leq x_1 \leq x_{1U}^{\alpha}$$

$$\beta_i = \min_{u_2} \|U\|$$

$$\text{s.t.} \qquad g_i(u_2) = 0$$

$$g_1(X) = x_1^2 * x_2/20 - 1$$

$$g_2(X) = (x_1 + x_2 - 5)^2/30 + (x_1 - x_2 - 12)^2/120 - 1$$

$$g_3(X) = 80\Big/\left(x_1^2 + 8 * x_2 + 5\right) - 1$$

$$\beta_{t_i} = 3, 3.2 \qquad \text{or} \qquad 3.3, \qquad i = 1 \sim 3$$

Table 6 compares the PBDO and RBDO results. For a target reliability index of three, the PBDO optimum is equal to 7.1804, which is higher than the RBDO optimum of 6.7257. For increasing reliability index values, the PBDO optimum also increases, as expected. For the deterministic, reliability, and possibility optima, only the first two constraints are active.

VII. Summary

The possibility theory was used to assess design reliability with incomplete information. The possibility theory was viewed as a variant of fuzzy set theory. The different types of uncertainty and formal uncertainty theories, including the possibility theory, were first introduced using the fundamentals of fuzzy measures. Subsequently, the numerical methods for propagating nonprobabilistic uncertainty were described because they provide the foundation of possibility-based design.

The commonly used vertex and discretization methods were reviewed and compared with a proposed hybrid (global–local) optimization method through examples. The input uncertainties were represented using convex normal membership functions. We showed that the vertex method is computationally efficient, but it can be inaccurate if the transfer function exhibits maxima or minima within the domain defined by the extreme values of the input variables. The discretization method is accurate but can be prohibitively expensive because the number of function evaluations increases exponentially with the number of input variables. For these reasons, a computationally efficient and accurate hybrid (global–local) optimization approach was developed for calculating the confidence level of fuzzy response. The DIRECT global optimizer is first used, followed by a local optimizer based on sequential quadratic programming. A variety of examples demonstrated that the hybrid optimization method is very efficient and has the same accuracy with the discretization method.

The possibility theory was also used in design. A possibility-based design optimization method in which all design constraints are expressed possibilistically was proposed. It was shown that the method gives a conservative solution compared with all conventional reliability-based designs obtained with different probability distributions. A general possibility-based design optimization method was also presented, which handles a combination of random and possibilistic design variables. Several numerical examples demonstrated the application of possibility theory in design.

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